Transmission Equation for Multiple Cooperative Transmitters and Collective Beamforming

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***Abstract*—A general transmission equation is derived for multiple simultaneous transmitters of arbitrary phase coherence. A closed form result is obtained by examining the expected value of the power at the output of the receive antenna. The applicability of the results to collective beamforming is also discussed.**

***Index Terms*—Beamforming, distributed array, Friis equation.**

I. Introduction

THERE are many applications in radar, communications and electronic warfare (EW) where collective transmissions from several individual transmitters can improve performance over that of a single one. One military example is the use of a large number of small relatively low power jammers on unmanned air vehicles (UAVs). A swarm of small UAVs could operate at a closer range to the victim receiver with less risk of detection than a large aircraft carrying a high power jammer. The UAVs can be made low cost and expendable, and a small number of UAV losses can be tolerated without degrading the capability of the collective group.

A large number of low power radar or communication system transmitters can potentially provide the same power as a larger, heavier and more expensive transmitter. The maximum effectiveness is achieved when the transmitters are synchronized to a common reference in time and frequency. The synchronization of local oscillators (LOs) that are widely distributed in space (relative to carrier wavelength λ) is a difficult problem. However it is currently the topic of much research due to its crucial role in collective beamforming and distributed sensor systems [1]–[7].

This note presents a simple closed-form expression for the power at the receiver antenna output from multiple transmit signals as a function of random phase errors of the type that would occur with imperfect synchronization. However, the phase errors need not be limited to LO error, but can be due to any independent source of phase error between transmitters. An example is erroneous position data that would lead to incorrect phase weights when used in beamformer processing. Another example is transmission path phase errors arising from obstructions.

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II. General Case

Consider the time-harmonic case where phasor quantities are used and the time dependence is $e^{j\omega t}$. Let there be N transmitters at locations $(x_n, y_n, z_n), n = 1, \ldots, N$ with the $z = 0$ plane being the Earth's surface. The time-averaged transmit powers are P_{t_n} and the antenna gains $G_{t_n}(\theta_{t_n}, \phi_{t_n})$, where $(\theta_{t_n}, \phi_{t_n})$ are the standard spherical polar variables giving the direction to the targeted receive antenna from transmitter n. The target receiver is located at the origin at height z_r and has an antenna gain $G_r(\theta_n, \phi_n)$ in the direction of transmitter n.

Since the voltage from multiple transmitters must be added at the receive antenna output, it is convenient to use vector effective height (effective length), which is related to gain by [8] and [9]

$$\vec{h}_r(\theta_n, \phi_n) = 2\sqrt{\frac{G_r(\theta_n, \phi_n) R_a \lambda^2}{4\pi\eta_o}}\hat{e}_r. \quad (1)$$

The vector direction is determined by the antenna polarization $\hat{e}_r$. The receive antenna impedance is $Z_a = R_a + jX_a$ and η_o the impedance of free space.

Effective height relates the open circuit voltage at the antenna terminals to the incident electric field intensity. As shown in Fig. 1, the signal from the n th transmitter gives a voltage

$$V_{L_n} = \frac{1}{2}\vec{h}_r(\theta_n, \phi_n) \cdot \vec{E}_n^i. \quad (2)$$

For a conjugate matched condition, the total voltage across the receiver (a resistor R_L equal to the antenna resistance R_a) is

$$V_L = \sum_n V_{L_n} = \frac{1}{2}\sum_n \vec{h}_r(\theta_n, \phi_n) \cdot \vec{E}_n^i. \quad (3)$$

The time-averaged power density S_n^i at the receive antenna, which is at a distance $R_n = \sqrt{x_n^2 + y_n^2 + (z_n - z_r)^2}$ from the n th transmitter, is

$$S_n^i = \frac{\left|\vec{E}_n^i\right|^2}{2\eta_o} = \frac{P_{t_n} G_{t_n}(\theta_{t_n}, \phi_{t_n})}{4\pi R_n^2}. \quad (4)$$

Thus the incident electric field vector can be written as

$$\vec{E}_n^i = \hat{e}_n \sqrt{\frac{2\eta_o P_{t_n} G_{t_n}(\theta_{t_n}, \phi_{t_n})}{4\pi R_n^2}} \exp(-jkR_n + j\psi_n) \quad (5)$$

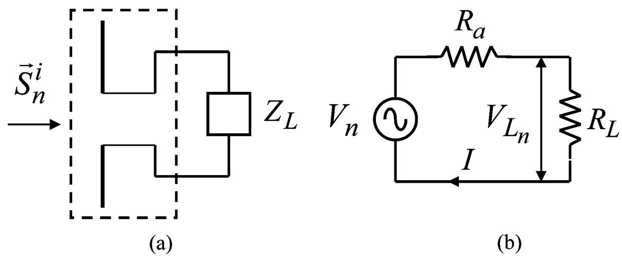

Fig. 1. (a) Receiving element and (b) its equivalent circuit when conjugate matched.

where $k = 2\pi/\lambda$ and $\hat{e}_n$ is defined by the polarization of the n th transmit antenna. The phase ψ_n is the sum of all phases other than free space path delay, and can include phase weights introduced for synchronization, errors due to LO variations, excess path phase from obstacles, etc. From (3) and (5) the total voltage out of the receive antenna is

$$V_L = \frac{1}{2}\sum_n \left\{ \frac{\sqrt{2P_{t_n} G_n(\theta_{t_n}, \phi_{t_n}) G_r(\theta_n, \phi_n) R_a \lambda^2}}{4\pi R_n} \times (\hat{e}_r \cdot \hat{e}_n) \exp(-jkR_n + j\psi_n) \right\} \quad (6)$$

and the power to the receiver is

$$P_L = |V_L|^2/(2R_L). \quad (7)$$

This is a general result that applies to any configuration of transmitters. In the next two sections special cases are considered: 1) the ideal case where there are no errors and 2) random phase errors.

III. Special Case of Identical Transmitters

Consider the special case where the following conditions apply:

1) Identical transmit powers and omnidirectional gains are used$P_{t_n} \approx P_t, G_{t_n}(\theta_{t_n}, \phi_{t_n}) \approx G_t$.
2) A common LO reference is used and exact phase shifts are added to focus all transmitters at the receive antenna: $kR_n + \psi_n \approx \text{constant for all } n$.
3) Let all antennas be vertical linear polarization so that $\hat{e}_r = \hat{z}$ and $\hat{e}_n = \hat{z}$.
4) All transmitters are in the same direction as viewed from the receive antenna: $G_r(\theta_n, \phi_n) \approx G_r$.
5) All ranges are approximately the same: $1/R_n \approx 1/R$.

Conditions 4 and 5 would exist if all of the transmitters were clustered at a sufficiently far range from the receive antenna.

With these assumptions the total voltage reduces to

$$|V_L| = N\sqrt{\frac{2R_a P_t G_t G_r \lambda^2}{(4\pi R)^2}}. \quad (8)$$

When inserted in (7) it gives the conventional Friis result for a total power transmitted NP_t and array gain NG_t.

IV. Effect of Random Errors

If the local oscillators are not synchronized, or the phase corrections are in error due to position uncertainty, then assumption 2 no longer holds. The effectiveness of operating collectively with synchronization or position errors is illustrated by letting ψ_n be a random variable, and computing the expected value of the power. Thus the voltage in (6) is dependent on a single random variable. However, this random variable could be the sum of several other independent random variables. In this way multiple independent error sources can be included in the calculation.

The mean power is

$$\langle P_L \rangle = \frac{\langle V_L V_L^* \rangle}{2R_L} = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2} \times \left\langle \left(\sum_n \frac{e^{-jkR_n + j\psi_n}}{R_n} \right) \left(\sum_m \frac{e^{-jkR_m + j\psi_m}}{R_m} \right)^* \right\rangle \quad (9)$$

where * denotes complex conjugate. For a large number of transmitters with independent uniformly distributed phases $[-\pi, \pi]$, which in the limit become Gaussian distributed,

$$\langle P_L \rangle = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2} \sum_{m,n} \frac{e^{-jk(R_n - R_m) + j\psi_m}}{R_m R_n} \left\langle e^{j(\psi_n - \psi_m)} \right\rangle. \quad (10)$$

A new random variable $\delta = \psi_n - \psi_m$ can be defined with variance $\overline{\delta^2}$. The expected value of the exponential is [10]

$$\langle \exp(j\overline{\delta^2}) \rangle = \langle \cos(\overline{\delta^2}) \rangle + j\langle \sin(\overline{\delta^2}) \rangle = \begin{cases} 1, & m = n \\ e^{-\overline{\delta^2}}, & m \neq n. \end{cases} \quad (11)$$

Using (11) in (10)

$$\langle P_L \rangle = \frac{P_t G_t G_r \lambda^2}{(4\pi)^2} \times \left\{ e^{-\overline{\delta^2}} \sum_{\substack{m,n \\ m \neq n}} \frac{e^{-jk(R_n - R_m)}}{R_m R_n} + \sum_{\substack{m,n \\ m=n}} \frac{1}{R_m R_n} \right\}. \quad (12)$$

Now add and subtract the sum

$$e^{-\overline{\delta^2}} \sum_{\substack{m,n \\ m=n}} \frac{e^{-jk(R_n - R_m)}}{R_m R_n}. \quad (13)$$

Rearranging and setting all ranges to R from condition 5 gives:

$$\langle P_L \rangle = \frac{P_t G_t G_r \lambda^2}{(4\pi R)^2} \left\{ N^2 e^{-\overline{\delta^2}} + \left(1 - e^{-\overline{\delta^2}}\right) N \right\}. \quad (14)$$

The first term in the brackets is the coherent part which increases as N^2, whereas the second term is the noncoherent part which increases as N. This is analogous to the result derived by Ruze [10] for the effect of random errors on antenna patterns.

Fig. 2 compares a typical Monte Carlo simulation (100 trials) and (14) for a rms error of 40°. Unit gains, power, and wavelength are used, and the range is 1000 m. The upper coherent limit and lower noncoherent limit are also shown for reference.

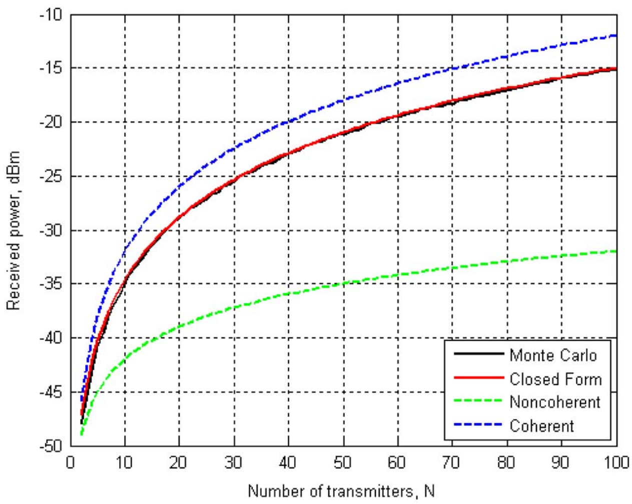

Fig. 2. Comparison of (13) with an average of 100 Monte Carlo simulations with unit gains, powers and wavelength, at a range of 1000 m, rms error is 40°.

V. Distributed Beamforming

Recently there have been proposed applications of distributed sensor systems that use collective or distributed beamforming [1], [2], [6]. For example, a large number of inexpensive transmitter/receiver units could be dropped over an open area. They would then self survey and self synchronize to a common time and frequency reference, and form a digital phased array system operated by a master controller. For coherent beamforming, the elements must know their position relative to a common reference so that they can compensate for range differences with a phase shift (i.e., focus the beam).

Fig. 3 shows the displacement of a transmitting element from its desired location. For element m, the far-field phase error introduced in the direction (θ,ϕ) by the change in path length is

$$\psi_m = k\{\Delta x_m \sin\theta\cos\phi + \Delta y_m \sin\theta\sin\phi + \Delta z_m \cos\theta\} \tag{15}$$

where $(\Delta x_m, \Delta y_m, \Delta z_m)$ are the displacement errors in the Cartesian directions. Often these errors can be modeled as random, in which case ψ_m in (15) becomes the random variable used in (9). If the errors are independent and zero mean with variances $(\overline{\delta_x^2}, \overline{\delta_y^2}, \overline{\delta_y^2})$ then

$$\overline{\delta^2} = k^2\left\{\overline{\delta_x^2}(\sin\theta\cos\phi)^2 + \overline{\delta_y^2}(\sin\theta\sin\phi)^2 + \overline{\delta_y^2}\cos^2\theta\right\}. \tag{16}$$

VI. Summary

A simple closed form expression for the power received from multiple simultaneous transmitters with random relative phase errors is presented. The case of identical transmitters with no phase error is equivalent to synchronizing all local oscillators to a common phase reference and compensating for phase differences due to the path lengths (i.e., focusing). In this case the total power increases with the number of transmitters as N^2. As the phase error increases to the point of being completely random, as would be the case for noise jammers, the total power increases as N.

Location error is an important concern in practical distributed systems operating coherently. For example, knowing the location of a UAV to within a fraction of a wavelength is unlikely, especially at frequencies above a gigahertz. Equation (14) can be used to determine the system degradation or, conversely, the maximum allowable position error.

The variation of N with phase error indicates that for many applications, adequate transmitter power can be achieved by simply increasing their number, without having to deal with the complexity of synchronizing a swarm of these transmitters.

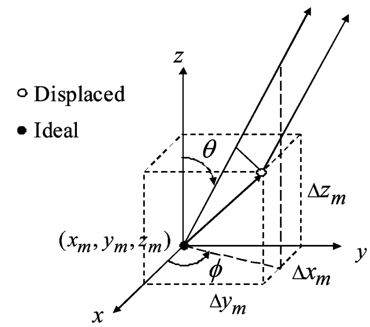

Fig. 3. Displacement of a transmitting element from its ideal location.

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